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Bosonization of vertex operators for the Z_n-symmetric Belavin model

Heng Fan‡, Bo-yu Hou‡, Kang-jie Shi†‡ and Wen-li Yang†‡

† CCAST (World Laboratory), PO Box 8730, Beijing 100080, China

‡ Institute of Modern Physics, Northwest University, Xian 710069, China§

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Abstract. Based on the bosonization of vertex operators for the $A_{n-1}^{(1)}$ face model by Asai, Jimbo, Miwa and Pugai, using vertex-face correspondence we obtain vertex operators for the Z_n-symmetric Belavin model, which are constructed by deformed boson oscillators.

1. Introduction

The Bosonization of vertex operators for solvable models is well known to be a very powerful way of studing their correlation functions [1–3, 28]. These vertex operators realize the Zamolodchikov–Faddeev algebras [4–7] with the R-matrices of the models.

Recently, the studies of q-deformed Virasoro algebra [8, 21, 22] and its vertex operators have made it clearer to understand the correspondence between the unitary minimal conformal models [26] and the ABF models [11]. This has been considered as a mystery for a long time. The bosonization for q-deformed Virasoro algebra and its vertex operators also make it possible to calculate the correlation functions for ABF models. The bosonization for q-deformed W algebra [21, 22] and its vertex operations [10, 21] encourage the investigation of the $A_{n-1}^{(1)}$ RSOS models [15]. Naively, the q-deformed W algebra (including q-deformed Virasoro algebra) would play an important role in the elliptic face models. How about the elliptic-vertex model, eight-vertex model and Z_n Belavin model? Jimbo *et al* [20] obtained the difference equations for eight-vertex operators using the method of the corner transfer matrix (CTM) and the method of 'physical picture', moreover, the spontaneous polarization of the eight-vertex model was also obtained. Quano [19] obtained, by using the same method, the difference equations for the Z_n Belavin model and the corresponding spontaneous polarization. However, to calculate general correlation functions for the elliptic vertex model practically is very complicated and is still an open problem [24, 27]. But the other effective way to solve these difference equations is to realize the vertex operators in terms of bosonic free fields [3, 8, 9, 24, 27, 29].

After the papers on trigonometric models [1–3] several important works for elliptic models have been done [8–10]. Lukyanov *et al* [8] gave the bosonization of vertex operators for the ABF model. Miwa and Weston [9] gave the corresponding bosonized boundary operators. Recently, Asai *et al* [10] obtained the bosonized vertex operators for the $A_{n-1}^{(1)}$ face model [12]. These works will greatly promote the study of solvable models of the elliptic type. Based on their work, using vertex-face correspondence [13–15], we obtain

§ Mailing address.

bosonized vertex operators for the Z_n-symmetric Belavin model [16, 17]. The construction of vertex operators and its dual is the main result of our paper.

In sections 2 and 3 we find an intertwiner which intertwines the Belavin R-matrix and the Boltzmann weight of the $A_{n-1}^{(1)}$ face model in [10]. In section 4 we then review the vertex operators in [8-10] and their exchange relations. In section 5, by combining their vertex operators and the intertwiners, we finally obtain the bosonization for vertex operators of the Z_n-symmetric Belavin model.

2. Vertex-face correspondence

Given an integer n ($2 \leqslant n$), and two complex numbers τ and w ($\mathrm{Im}\,\tau > 0$), we can construct the Z_n-symmetric Belavin R-matrix [16, 17]. Define $n \times n$ matrices g, h and I_α

$$g_{jk} = \omega^j \delta_{jk} \qquad h_{jk} = \delta_{j+1,k} \qquad \omega = \exp\left(\frac{2\mathrm{i}\pi}{n}\right)$$
$$I_\alpha = I_{(\alpha_1,\alpha_2)} = g^{\alpha_2} h^{\alpha_1} \qquad (\alpha_1, \alpha_2) \in Z_n^2.$$

Define $I_\alpha^{(j)} = I \otimes I \otimes \ldots \otimes I_\alpha \otimes I \otimes \ldots \otimes I$, where I_α is at the jth site, I is the $n \times n$ unit matrix, and

$$W_\alpha(z,\tau) = \frac{1}{n}\theta\begin{bmatrix} \frac{1}{2}+\frac{\alpha_1}{n} \\ \frac{1}{2}+\frac{\alpha_2}{n} \end{bmatrix}\left(z + \frac{w}{n}, \tau\right) \Big/ \theta\begin{bmatrix} \frac{1}{2}+\frac{\alpha_1}{n} \\ \frac{1}{2}+\frac{\alpha_2}{n} \end{bmatrix}\left(\frac{w}{n}, \tau\right)$$
$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{m \in Z} \exp\{\mathrm{i}\pi(m+a)[(m+a)\tau + 2(z+b)]\}$$
$$\sigma_0(z,\tau) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z,\tau).$$

The Z_n-symmetric Belavin R-matrix is

$$\bar{R}_{jk}(z,\tau) = \frac{\sigma_0(w,\tau)}{\sigma_0(z+w,\tau)} \sum_{\alpha \in Z_n^2} W_\alpha(z,\tau) I_\alpha^{(j)} (I_\alpha^{-1})^{(k)} \tag{1}$$

which satisfies the Yang–Baxter equation (YBE)

$$\begin{aligned} &R_{12}(z_1 - z_2, \tau) R_{13}(z_1 - z_3, \tau) R_{23}(z_2 - z_3, \tau) \\ &\quad = R_{23}(z_2 - z_3, \tau) R_{13}(z_1 - z_3, \tau) R_{12}(z_1 - z_2, \tau). \end{aligned} \tag{2}$$

Given an n-vector $a \in Z^n$, we define the Boltzmann weight of the $A_{n-1}^{(1)}$ face model [12], which can be written in the vertex form $\bar{W}(a|z,\tau)_{\mu\nu}^{\mu'\nu'}$, whose non-zero elements are

$$\bar{W}(a|z,\tau)_{\mu\mu}^{\mu\mu} = 1 \tag{3}$$

$$\bar{W}(a|z,\tau)_{\nu\mu}^{\mu\nu} = \frac{\sigma_0(z+w,\tau)}{\sigma_0(w,\tau)} \frac{\sigma_0(z+a_{\mu\nu}w,\tau)}{\sigma_0(a_{\mu\nu}w,\tau)} \qquad \mu \neq \nu \tag{4}$$

$$\bar{W}(a|z,\tau)_{\mu\nu}^{\mu\nu} = \frac{\sigma_0(z,\tau)\sigma_0(a_{\mu\nu}w - w,\tau)}{\sigma_0(z+w,\tau)\sigma_0(a_{\mu\nu}w,\tau)} \qquad \mu \neq \nu \tag{5}$$

where $a_{\mu\nu}$ is defined by $a = (a_1, \ldots, a_n)$

$$\bar{a}_\mu = a_\mu - \frac{1}{n}\sum_{l=1}^{n} a_l + w_\mu \tag{6}$$

$$a_{\mu\nu} = \bar{a}_\mu - \bar{a}_\nu = a_\mu - a_\nu + w_\mu - w_\nu \tag{7}$$

where $\{w_j\}$ is a set of generic complex numbers specified by the face model under investigation.

The intertwiners of the vertex-face correspondence are [14, 15] n-column vectors $\varphi_{\mu,a}(z,\tau)$ whose kth component is

$$\varphi^{(k)}_{\mu,a}(z,\tau) = \theta^{(j)}\left(z + nw\left(\bar{a}_\mu + 1 - \frac{1}{n}\right), \tau\right)$$
$$\theta^{(j)}(z,\tau) = \theta\begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{bmatrix}(z, n\tau).$$

The vertex-face correspondence is

$$\bar{R}(z_1 - z_2, \tau)\varphi_{\mu,a+e_\nu}(z_1,\tau) \otimes \varphi_{\nu,a}(z_2,\tau)$$
$$= \sum_{\mu'\nu'} \bar{W}(a|z_1 - z_2, \tau)^{\mu\nu}_{\mu'\nu'} \varphi_{\mu',a}(z_1,\tau) \otimes \varphi_{\nu',a+e_{\mu'}}(z_2,\tau) \tag{8}$$

where $e_\mu = (0, 0, \ldots, 1, 0, \ldots, 0)$ and '1' is at the μth site. We can introduce n-row vectors $\tilde{\varphi}_{\mu,a}(z,\tau)$ such that

$$\sum_k \tilde{\varphi}^{(k)}_{\mu,a}(z,\tau)\varphi^{(k)}_{\nu,a}(z,\tau) = \delta_{\mu\nu}. \tag{9}$$

Thus we have

$$\sum_\mu \varphi_{\mu,a}(z,\tau)\,\tilde{\varphi}_{\mu,a}(z,\tau) = I. \tag{10}$$

Note that $\tilde{\varphi}^{(k)}_{\mu,a}(z,\tau)$ is a function of $a, \mu, k, z, n, \tau, w, \{w_j\}$. One can show the vertex-face correspondence by $\tilde{\varphi}_{\mu,a}$

$$\tilde{\varphi}_{\mu,a}(z_1,\tau) \otimes \tilde{\varphi}_{\nu,a+e_\mu}(z_2,\tau)\bar{R}(z_1 - z_2,\tau)$$
$$= \sum_{\mu'\nu'} \bar{W}(a|z_1 - z_2,\tau)^{\mu'\nu'}_{\mu\nu}\,\tilde{\varphi}_{\mu',a+e_{\nu'}}(z_1,\tau) \otimes \tilde{\varphi}_{\nu',a}(z_2,\tau). \tag{11}$$

3. Modular transformation of Boltzmann weight

Since the Boltzmann weight in equations (3)–(5) is not the same as that in [10], we need to rescale the Z_n Belavin R-matrix and specify the parameters z and τ, so that we can directly use the beautiful results of [10]. Let us restrict the parameter w : Im $w > 0$ and set

$$x = \mathrm{e}^{\mathrm{i}\pi w} \qquad |x| < 1 \qquad (\mathrm{Im} w > 0) \qquad n + 2 \leqslant r$$
$$[v] = \mathrm{e}^{\mathrm{i}\pi\frac{wv^2}{r}}\sigma_0(wv, rw) = \text{constant} \times x^{\frac{v^2}{r} - v}\Theta_{x^{2r}}(x^{2v})$$
$$\Theta_q(z) = (z,q)(qz^{-1},q)(q,q) \qquad (z,q) = \prod_{n=0}^{\infty}(1 - zq^n).$$

Note that the modular transformation for the theta function $\sigma_0(z,\tau)$

$$\sigma_0\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \text{constant} \times \mathrm{e}^{\frac{\mathrm{i}\pi z^2}{\tau}}\sigma_0(z,\tau) \tag{12}$$

we have

$$[v] = \text{constant} \times \sigma_0\left(\frac{vw}{rw}, -\frac{1}{rw}\right). \tag{13}$$

Let the Z_n-symmetric Belavin R-matrix be rescaled, and the parameters z and τ be specified as follows

$$\begin{aligned}
&R\left(v,-\frac{1}{rw}\right)=r_1(-v)\frac{\sigma_0\left(\frac{1}{r},-\frac{1}{rw}\right)}{\sigma_0\left(\frac{v+1}{r},-\frac{1}{rw}\right)}\sum_\alpha W_\alpha\left(v,-\frac{1}{rw}\right)I_\alpha\otimes I_\alpha^{-1}\\
&W_\alpha\left(v,-\frac{1}{rw}\right)=\frac{\sigma_\alpha\left(\frac{v}{r}+\frac{1}{nr},-\frac{1}{rw}\right)}{n\sigma_\alpha\left(\frac{1}{nr},-\frac{1}{rw}\right)}\\
&r_1(v)=x^{2v\frac{(r-1)(n-1)}{nr}}\frac{g_1(-v)}{g_1(v)}\qquad g(v)=\frac{\{x^{2+2v}\}\{x^{2r+2n-2+2v}\}}{\{x^{2r+2v}\}\{x^{2n+2v}\}}\\
&\{z\}=(z;x^{2r},x^{2n})\qquad (z;q_1,q_2,\ldots,q_m)=\prod_{\{n_j\}=0}^{\infty}(1-zq_1^{n_1}q_2^{n_2}\ldots q_m^{n_m})
\end{aligned}\tag{14}$$

we also specify the intertwiners φ and $\tilde{\varphi}$ as follows

$$\begin{aligned}
&\varphi^{(k)}_{\mu,a}\left(v,-\frac{1}{rw}\right)=\theta^{(k)}\left(\frac{v+n(\bar{a}_\mu+1-\frac{1}{n})}{r},-\frac{1}{rw}\right)\\
&\sum_k\tilde{\varphi}^{(k)}_{\mu,a}\left(v,-\frac{1}{rw}\right)\varphi^{(k)}_{\nu,a}\left(v,-\frac{1}{rw}\right)=\delta_{\mu\nu}.
\end{aligned}\tag{15}$$

The vertex-face correspondence becomes

$$\begin{aligned}
&\tilde{\varphi}_{\mu,a}\left(v_1,-\frac{1}{rw}\right)\otimes\tilde{\varphi}_{\nu,a+e_\mu}\left(v_2,-\frac{1}{rw}\right)R\left(v_1-v_2,-\frac{1}{rw}\right)\\
&\quad=\sum_{\mu'\nu'}W\left(a|v_1-v_2,-\frac{1}{rw}\right)^{\mu'\nu'}_{\mu\nu}\tilde{\varphi}_{\mu',a+e_{\nu'}}\left(v_1,-\frac{1}{rw}\right)\otimes\tilde{\varphi}_{\nu',a}\left(v_2,-\frac{1}{rw}\right).
\end{aligned}\tag{16}$$

Using equation (12), the non-zero element of $W\left(a|v,-\frac{1}{rw}\right)^{\mu'\nu'}_{\mu\nu}$ can be written

$$W\left(a|v,-\frac{1}{rw}\right)^{\mu\mu}_{\mu\mu}=r_1(-v)\tag{17}$$

$$W\left(a|v,-\frac{1}{rw}\right)^{\mu\nu}_{\mu\nu}=r_1(-v)\frac{[v][a_{\mu\nu}-1]}{[v+1][a_{\mu\nu}]}\tag{18}$$

$$W\left(a|v,-\frac{1}{rw}\right)^{\mu\nu}_{\nu\mu}=r_1(-v)\frac{[v+a_{\mu\nu}][1]}{[v+1][a_{\mu\nu}]}.\tag{19}$$

It can be found that our Boltzmann weight $W\left(a|-v,-\frac{1}{rw}\right)^{\mu'\nu'}_{\mu\nu}$ is the same as the Boltzmann weight $W\begin{pmatrix}a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\mu\\ a+\bar{\epsilon}_\nu & a\end{pmatrix}|v\Big)$ in [10]

$$W\left(a|-v,-\frac{1}{rw}\right)^{\mu\mu}_{\mu\mu}=r_1(v)=W\begin{pmatrix}a+2\bar{\epsilon}_\mu & a+\bar{\epsilon}_\mu\\ a+\bar{\epsilon}_\mu & a\end{pmatrix}|v\Big)\tag{20}$$

$$W\left(a|-v,-\frac{1}{rw}\right)^{\mu\nu}_{\mu\nu}=r_1(v)\frac{[v][a_{\mu\nu}-1]}{[v-1][a_{\mu\nu}]}=W\begin{pmatrix}a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\mu\\ a+\bar{\epsilon}_\nu & a\end{pmatrix}|v\Big)\tag{21}$$

$$W\left(a|-v,-\frac{1}{rw}\right)^{\mu\nu}_{\nu\mu}=r_1(v)\frac{[v-a_{\mu\nu}][1]}{[v-1][a_{\mu\nu}]}=W\begin{pmatrix}a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\nu\\ a+\bar{\epsilon}_\nu & a\end{pmatrix}|v\Big).\tag{22}$$

4. Vertex operators in the $A^{(1)}_{n-1}$ face model

We review Asai *et al's* [10] bosonization of vertex operators for the $A^{(1)}_{n-1}$ face model.

According to [8–10] we introduce bosonic oscillators $\beta^j_m (1 \leqslant j \leqslant n-1, m \in Z\backslash\{0\})$ which satisfy

$$[\beta^j_m, \beta^j_{m'}] = m\frac{[(n-1)m]_x[(r-1)m]_x}{[nm]_x[rm]_x}\delta_{m+m',0} \tag{23}$$

$$[\beta^j_m, \beta^k_{m'}] = -mx^{\mathrm{sgn}(j-k)nm}\frac{[m]_x[(r-1)m]_x}{[nm]_x[rm]_x}\delta_{m+m',0} \qquad j \neq k$$
$$[a]_x = \frac{x^a - x^{-a}}{x - x^{-1}} \qquad x = \mathrm{e}^{\mathrm{i}\pi w}. \tag{24}$$

Define $\beta^n_m = -x^{2mn}\sum_{j=1}^{n-1} x^{-2jm}\beta^j_m$.

Introduce zero modes p_μ, $q_\mu (\mu = 1, \ldots, n)$, such that $[\mathrm{i}p_\mu, q_\nu] = \delta_{\mu,\nu}$. Consider orthonormal bases $\{e_\mu\}$, $\mu = 1, \ldots, n \langle e_\mu, e_\nu\rangle = \delta_{\mu\nu}$, and set

$$\bar{e}_\mu = e_\mu - \frac{1}{n}\sum_k e_k.$$

Define

$$Q_{\bar{e}_\mu} = q_\mu - \frac{1}{n}\sum_k q_k \qquad P_{\bar{e}_\mu} = p_\mu - \frac{1}{n}\sum_k p_k.$$

One has

$$[\mathrm{i}P_{\bar{e}_\mu}, Q_{\bar{e}_\nu}] = \delta_{\mu\nu} - \frac{1}{n} = \langle \bar{e}_\mu, \bar{e}_\nu\rangle.$$

Let the vacuum $|0\rangle$ be such that

$$\beta^j_m|0\rangle = p_\mu|0\rangle = 0 \qquad \text{for } m > 0$$

and that

$$|l, k\rangle = \mathrm{e}^{\mathrm{i}\sqrt{\frac{r}{r-1}}Q_l - \mathrm{i}\sqrt{\frac{r-1}{r}}Q_k}|0\rangle$$

where $l = \sum_{j=1} l_j\bar{e}_j$, $k = \sum_{j=1} k_j\bar{e}_j$ and $\{l_j\}$, $\{k_j\}$ are integers. Let

$$\gamma = \gamma_j e_j \qquad \beta = \sum_j \beta_j e_j \qquad P_k = \sum_j k_j P_{\bar{e}_j}$$

we have

$$[\mathrm{i}P_\gamma, Q_\beta] = \langle\overline{\gamma}, \beta\rangle = \langle\gamma, \overline{\beta}\rangle$$

where $\overline{\gamma} = \sum_j \gamma_j\bar{e}_j$ $\overline{\beta} = \sum_j \beta_j\bar{e}_j$. The Fock space $F_{l,k} = C[\{\beta^j_{-1}, \beta^j_{-2}, \ldots\}_{1\leqslant j\leqslant n}]|l, k\rangle$ with

$$\beta^j_m|l, k\rangle = 0 (m > 0)$$
$$P_\gamma|l, k\rangle = \left\langle\overline{\gamma}, \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k\right\rangle. \tag{25}$$

For $j = 1, \ldots, n-1$ define:

$$\text{the simple root } \alpha_j = e_j - e_{j+1} \qquad \text{the basic weight } \omega_j = \sum_{k=1}^{j}\bar{e}_k$$

$$\xi_j(v) = \mathrm{e}^{\mathrm{i}\sqrt{\frac{r-1}{r}}(Q_{\alpha_j} - \mathrm{i}P_{\alpha_j}2v\ln x)}\mathrm{e}^{\sum_{m\neq 0}\frac{1}{m}(\beta^j_m - \beta^{j+1}_m)x^{-(j+2v)m}}$$

$$\eta_j(v) = \mathrm{e}^{-\mathrm{i}\sqrt{\frac{r-1}{r}}(Q_{\omega_j} - \mathrm{i}P_{\omega_j}2v\ln x)}\mathrm{e}^{-\sum_{m\neq 0}\frac{1}{m}\sum_{k=1}^{j}\beta^k_m x^{(j-2k+1-2v)m}}.$$

Introduce vertex operators

$$\begin{aligned}\phi_\mu(v) &= \oint \prod_{j=1}^{\mu-1} \frac{\mathrm{d}(x^{2v_j})}{2\mathrm{i}\pi x^{2v_j}} \eta_1(v)\xi_1(v_1)\ldots\xi_{\mu-1}(v_{\mu-1}) \prod_{j=1}^{\mu-1} f(v_j - v_{j-1}, \hat{\pi}_{j\mu}) \\ \hat{\pi}_\mu &= \sqrt{r(r-1)}P_{\bar{e}_\mu} + w_\mu f(v, y) = \frac{[v+\frac{1}{2}-y]}{[v-\frac{1}{2}]}\end{aligned} \tag{26}$$

where $\{w_j\}$ is a set complex number defined in equation (6) and $\hat{\pi}_\mu$ is a set of operators. Here we set $v_0 = v$ and take the integration contours to be simple closed curves around the origin satisfying

$$|xx^{2v_{j-1}}| < |x^{2v_j}| < |x^{-1}x^{2v_{j-1}}| \qquad (j = 1, \ldots, \mu - 1).$$

Following [10], one can verify

$$\phi_\mu(v_1)\phi_\nu(v_2) = \sum_{\mu'\nu'} \phi_{\nu'}(v_2)\phi_{\mu'}(v_1) W\left(\hat{\pi}|v_2 - v_1, -\frac{1}{rw}\right)_{\mu'\nu'}^{\mu\nu}.$$

Namely,

$$\phi_\mu(-v_1)\phi_\nu(-v_2) = \sum_{\mu'\nu'} \phi_{\nu'}(v_2)\phi_{\mu'}(v_1) W\left(\hat{\pi}|v_1 - v_2, -\frac{1}{rw}\right)_{\mu'\nu'}^{\mu\nu}. \tag{27}$$

Now the Boltzmann weight $W\left(\hat{\pi}|v, -\frac{1}{rw}\right)_{\mu'\nu'}^{\mu\nu}$ is some functions like equations (17)–(19) with $a_{\mu\nu}$ replaced by the operator $\hat{\pi}_{\mu\nu}$. Thus, it does not commutate with the vertex operator $\phi_\mu(v)$ and the exchange relations equation (27) should be written in that order.

5. Vertex operators for the Z_n-symmetric Belavin model

In the 'physical picture' of lattice models [1, 3, 15, 20], the vertex operators for the Z_n-symmetric Belavin model can be realized by a half-column transfer matrix, and these vertex operators realize the Zamolodchikov–Faddeev algebra with the Z_n-symmetric R-matrix as its construction coefficent [19, 20, 29]

$$Z^j(z_1)Z^k(z_2) = \sum_{j'k'} Z^{k'}(z_2)Z^{j'}(z_1)R_{j'k'}^{jk}(z_1 - z_2, \tau) \tag{28}$$

where the R-matrix is defined in equation (14) and $Z^j(z)$ are some operators acting on the eigenvaluevector spaces $\boldsymbol{H}^i$ of the CTM's Hamiltonian $D^{(i)}$ [3, 13]. The correlation functions are expressed in terms of the trace of vertex operators in the spaces $\boldsymbol{H}^i$. For more detail, refer to [3].

Our main idea is to realize these vertex operators in a direct sum of Fock space $L_i = \oplus_{\{m_i\}\in Z} F_{l,\overline{\Lambda}_i+\sum_{j=1}^{n-1} m_j\alpha_j}$, where $\overline{\Lambda}_i$ (resp. α_i) is the basic weight of Lie algebra A_{n-1} (resp. the simple root of Lie algebra A_{n-1}). This representation is expected to be irreducible for the generic r.

Define $\hat{a}_\mu = -\sqrt{\frac{r}{r-1}}p_\mu + w_\mu + \frac{r}{r-1}\langle e_\mu, l\rangle \hat{a}_{\mu\nu} = \hat{a}_\mu - \hat{a}_\nu$, we have

$$\begin{aligned}&\hat{a}_\mu \mathrm{e}^{-\mathrm{i}\sqrt{\frac{r-1}{r}}Q_\nu} = \mathrm{e}^{-\mathrm{i}\sqrt{\frac{r-1}{r}}Q_\nu}(\hat{a}_\mu + \delta_{\mu\nu}) \\ &\hat{\pi}_{\mu\nu}F_{l,k} = (r\langle e_\mu - e_\nu, l - k\rangle + \langle e_\mu - e_\nu, k\rangle + w_\mu - w_\nu)F_{l,k} \\ &\hat{a}_{\mu\nu}F_{l,k} = (\langle e_\mu - e_\nu, k\rangle + w_\mu - w_\nu)F_{l,k}.\end{aligned} \tag{29}$$

From the above equation and $[v+r]=-[v]$, we can derive

$$W\left(\hat{\pi}|v,-\frac{1}{rw}\right)_{\mu'\nu'}^{\mu\nu}|_{F_{l,k}}=W\left(\hat{a}|v,-\frac{1}{rw}\right)_{\mu'\nu'}^{\mu\nu}|_{F_{l,k}}. \tag{30}$$

From equations (11) and (27), we can construct the bosonization for the vertex operators of the Z_n-symmetric Belavin model which satisfy relation equation (28) by specifying the parameters z and τ. Define

$$\Phi^{(j)}(v)=\sum_{\mu}\phi_{\mu}(-v)\,\overset{\sim(j)}{\varphi}_{\mu,\hat{a}}\left(v+\delta,-\frac{1}{rw}\right) \tag{31}$$

where δ is a generic parameter. Note that equations (27) and (31) and the vertex-face correspondence equation (11), have

$$\begin{aligned}
\Phi^{(i)}(v_1)\Phi^{(j)}(v_2)|_{F_{l,k}} &= \sum_{\mu\nu}\phi_{\mu}(-v_1)\,\overset{\sim(i)}{\varphi}_{e_\mu,\hat{a}}\left(v_1+\delta,-\frac{1}{rw}\right)\\
&\quad\times\phi_\nu(-v_2)\,\overset{\sim(j)}{\varphi}_{e_\nu,\hat{a}}\left(v_2+\delta,-\frac{1}{rw}\right)|_{F_{l,k}}\\
&= \sum_{\mu\nu}\phi_{\mu}(-v_1)\phi_\nu(-v_2)\,\overset{\sim(i)}{\varphi}_{e_\mu,\hat{a}+e_\nu}\left(v_1+\delta,-\frac{1}{rw}\right)\overset{\sim(j)}{\varphi}_{e_\nu,\hat{a}}\left(v_2+\delta,-\frac{1}{rw}\right)|_{F_{l,k}}\\
&= \sum_{\mu\nu}\sum_{\mu'\nu'}\phi_{\nu'}(v_2)\phi_{\mu'}(v_1)W\left(\hat{\pi}|v_1-v_2,-\frac{1}{rw}\right)_{\mu'\nu'}^{\mu\nu}\overset{\sim(i)}{\varphi}_{e_\mu,\hat{a}+e_\nu}\left(v_1+\delta,-\frac{1}{rw}\right)\\
&\quad\times\overset{\sim(j)}{\varphi}_{e_\nu,\hat{a}}\left(v_2+\delta,-\frac{1}{rw}\right)|_{F_{l,k}}\\
&= \sum_{i'j'}\sum_{\mu'\nu'}\phi_{\nu'}(v_2)\phi_{\mu'}(v_1)\,\overset{\sim i'}{\varphi}_{e_{\mu'},\hat{a}}\left(v_1+\delta,-\frac{1}{rw}\right)\overset{\sim j'}{\varphi}_{e_\nu,\hat{a}+e_\mu}\left(v_2+\delta,-\frac{1}{rw}\right)\\
&\quad\times R_{i'j'}^{ij}\left(v_1-v_2,-\frac{1}{rw}\right)|_{F_{l,k}}.
\end{aligned}$$

Namely, we have the bosonization for vertex operators of the Z_n-symmetric Belavin model

$$\Phi^{(i)}(v_1)\Phi^{(j)}(v_2)|_{L_i}=\sum_{i'j'}\Phi^{(j')}(v_2)\Phi^{(i')}(v_1)R_{i'j'}^{ij}\left(v_1-v_2,-\frac{1}{rw}\right)|_{L_i}. \tag{32}$$

Moreover, the dual vertex operators $\Phi_j^*(v)$ are needed for construction. We define the dual-vertex operators $\Phi_\mu^*(v)$ through the skew-symmetric fusion of $n-1\Phi(v)$ [10]

$$\Phi_j^*(v)=\sum_{\mu}\overline{\phi}_\mu^{*(n-1)}\left(v-\frac{n}{2}\right)A_\mu^{-1}\varphi_{e_\mu,\hat{a}-e_\mu}^{j}\left(v,-\frac{1}{rw}\right)$$

$$A_\mu=(-1)^{n-1}\frac{1}{(x^2;x^{2r})(x^{2r-2};x^{2r})}\prod_{k=1}^{n}[1+\hat{\pi}_{k\mu}]$$

$$\overline{\phi}_\mu^{*(n-1)}(v)=\oint\prod_{j=\mu}^{n-1}\frac{\mathrm{d}(x^{2v_j})}{2\mathrm{i}\pi x^{2v_j}}\eta_{n-1}(v)\xi_{n-1}(v_{n-1})\ldots\xi_\mu(v_\mu)\prod_{j=\mu+1}^{n}f(v_{j-1}-v_j,\hat{\pi}_{\mu j})$$

where we set $v_n=v$ and $|xx^{2v_{j-1}}|<|x^{2v_j}|<|x^{-1}x^{2v_{j-1}}|$ $(j=\mu,\ldots,n-1)$.
Following equation (c.20) in [10] and equation (10), we have the following invertibility

$$\begin{aligned}
&\Phi^{(i)}(v)\Phi_j^*(v)|_{L_i}=c_n^{-1}\delta_j^i\times \mathrm{id}|_{L_i}\\
&c_n=x^{\frac{r-1}{r}\frac{n(n-1)}{2n}}\frac{g_{n-1}(x^n)}{(x^2;x^{2r})(x^{2r};x^{2r})^{2n-3}}\qquad g_{n-1}(z)=\frac{\{x^nz\}\{x^{2r+n}z\}}{\{x^{2r+n-2}z\}\{x^{n+2}z\}}.
\end{aligned} \tag{33}$$

6. Discussions

In this paper we construct the bosonic realization of vertex operators for the Z_n-symmetric Belavin model using vertex-face correspondence. Actually, this procedure from the vertex operators $\Phi_j(v)$ of the Z_n-symmetric Belavin model to the vertex operators $\phi_\mu(v)$ of the $A^{(1)}_{n-1}$ face model is a dynamic twisting procedure [30, 31], where the intertwining functions $\overset{\sim(k)}{\varphi}_{\mu,\hat{a}}$ which take the value on the operators, play an important role. In order to solve the correlation functions, we should construct the Hamiltonian D^i of CTM in the Fock space which has the following commutation relations with the vertex operators $\Phi_j(v)$ of the Z_n-symmetric Belavin model

$$\mathrm{e}^{\mathrm{i}n\pi wD}\Phi_j(v)\mathrm{e}^{-\mathrm{i}n\pi wD} = \Phi_j(v+n).$$

Unfortunately, this kind of operator D has not been found, but, it is well known that the $A^{(1)}_{n-1}$ face model is equivalent to the Z_n-symmetric Belavin model for the generic parameters x and r (or w and τ) [11, 13, 15]. Therefore, the correlation functions of the two equivalent model should be related to each other somehow. We will further study the relation between the two models. Fortunately, we can solve the correlation functions of the $A^{(1)_{n-1}}$ face model by the method of bosonization.

To solve the $A^{(1)}_{n-1}$ face model, we construct the operator D_F (which is the Hamiltonian of CTM in the $A^{(1)}_{n-1}$ face model) in the Fock space

$$D_F = \sum_{m=1}^{\infty}\sum_{j=1}^{n-1}\frac{[rm]_x}{[(r-1)m]_x}\Omega^j_{-m}S^j_m + \frac{1}{2}\sum_{j=1}^{n-1}P_{\omega_j}P_{\alpha_j} \tag{34}$$

$$\begin{aligned}
&\Omega^j_{-m} = \sum_{k=1}^{j} x^{(2k-j-1)m}\beta^k_{-m} \qquad S^j_m = x^{-jm}(\beta^j_m - \beta^{j+1}_m)\\
&D_F|_{L_i} = D^{(i)}_F|_{L_i}\\
&x^{nD_F}\phi_\mu(v)x^{-nD_F} = \phi_\mu(v+n).
\end{aligned} \tag{35}$$

The correlation function for the $A^{(1)}_{n-1}$ face model can be described by the following trace functions

$$F^{(i)}(v_1,\ldots,v_N)_{\mu_1,\ldots,\mu_N} = \frac{\mathrm{tr}_{L_i}(x^{nD_F}\phi_{\mu_1}(v_1)\ldots\phi_{\mu_N}(v_N)\phi^*_{\mu_N}(v_N)\phi^*_{\mu_{N-1}}(v_{N-1})\ldots\phi^*_{\mu_1}(v_1))}{\mathrm{tr}_{L_i}(x^{nD_F})}. \tag{36}$$

Using the cyclic properties of a matrix trace and the relations, equations (32) and (35), it is easy to derive the difference equations which the correlation functions should satisfy.

In the simple case for $N = 1$, the trace functions will give the character of Z-grade spaces L_i

$$\mathrm{tr}_{L_i}(x^{nD_F}) = \frac{\sum_{\{m_j\}\in Z} x^{\frac{1}{2}\sum_{k=1}^{n-1}\left\langle\omega_k,\sqrt{\frac{r}{r-1}}l-\sqrt{\frac{r-1}{r}}\left(\overline{\Lambda}_i+\sum_{j=1}^{n-1}m_j\alpha_j\right)\right\rangle\left\langle\alpha_k,\sqrt{\frac{r}{r-1}}l-\sqrt{\frac{r-1}{r}}\left(\overline{\Lambda}_i+\sum_{j=1}^{n-1}m_j\alpha_j\right)\right\rangle}}{(x^{2n};x^{2n})^{n-1}}. \tag{37}$$

Remark. Actually, the above character of space L_i is the same as that of the level one integrable representation of q-deformed affine algebra $(U_q(\hat{\mathrm{sl}(n)}))$, of course, it is also equal to that of the level one representation of affine algebra $(A^{(1)}_{n-1})$ [18]. Thus the space L_i would be some level one representation of some elliptic deformation of affine algebra [23], which are not known but many phenomena suggest that it would exist. We expect to find this elliptic deformation of affine algebra which would play a role of the symmetric algebra of the elliptic-type model.

For generic N, one will encounter the following trace functions

$$\mathrm{tr}_{L_i}(x^{nD_F}\mathrm{e}^{\sum_{m=1}^{\infty}\sum_{j=1}^{n-1}A_m^j\beta_{-m}^j}\mathrm{e}^{\sum_{m=1}^{\infty}\sum_{j=1}^{n-1}B_m^j\beta_m^j}f^{P_\gamma}). \tag{38}$$

Since the operator e^{Q_α} would shift a Fock sector to another different sector unless $Q_\alpha = 0$, the term of e^{Q_α} in general non-zero trace functions should be equal to 1. We can calculate the contributions in the trace for tensor components (i) oscillators modes and (ii) the zero mode separately. The trace over the oscillator part can be carried out by using the Clavelli–Shapiro technique [25]. More explicitly, let us introduce other oscillators $\overline{\beta}_m^j (j = 1, \ldots, n-1)$ which commute with the old ones β_m^j. Define the following operators acting in the tensor product of Fock space of β_m^j and that of $\overline{\beta}_m^j$

$$b_m^j = \frac{\beta_m^j \otimes 1}{1 - x^{2nm}} + 1 \otimes \overline{\beta}_{-m}^j \qquad m > 0$$

$$b_m^j = \beta_m^j \otimes 1 + \frac{1 \otimes \overline{\beta}_{-m}^j}{x^{2nm} - 1} \qquad m < 0.$$

Now the trace of some bosonic operator $\mathrm{O}(\beta_m^j)$ can be expressed in terms of the vacuum expectation value $\langle 0|\mathrm{O}(b_m^j)|0\rangle$. Namely,

$$\mathrm{tr}(x^{nD}\mathrm{O}(\beta_m^j)) = \frac{\langle 0|\mathrm{O}(b_m^j)|0\rangle}{(x^{2n}; x^{2n})}. \tag{39}$$

We denote $\langle 0|\mathrm{O}(b_m^j)|0\rangle$ by $\langle\langle \mathrm{O}(\beta_m^j)\rangle\rangle$ (we choose the same symbol as that of the Lukynov's in [8]). Due to the Wick theorem, the expectation value of a product of exponential operators is factorized into the two point functions

$$\langle\langle \eta_1(v_1)\eta_1(v_2)\rangle\rangle = \langle\langle \eta_{n-1}(v_1)\eta_{n-1}(v_2)\rangle\rangle C_1^2 G_1(v_2 - v_1) \tag{40}$$

$$\langle\langle \eta_1(v_1)\eta_{n-1}(v_2)\rangle\rangle = C_1^2 G_{n-1}(v_2 - v_1) \tag{41}$$

$$\langle\langle \eta_1(v_1)\xi_1(v_2)\rangle\rangle = C_1 C_2 S(v_2 - v_1) \tag{42}$$

$$\langle\langle \xi_j(v_1)\xi_{j+1}(v_2)\rangle\rangle = C_2^2 S(v_2 - v_1) \tag{43}$$

$$\langle\langle \xi_j(v_1)\xi_j(v_2)\rangle\rangle = C_2^2 T(v_2 - v_1) \tag{44}$$

$$\langle\langle \eta_{n-1}(v_1)\xi_{n-1}(v_2)\rangle\rangle = C_1 C_2 S(v_2 - v_1) \tag{45}$$

$$\{z\}' = (z; x^{2r}, x^{2n}, x^{2n}, x^{2n}) \qquad \rho(\eta_j) = C_1 = \frac{\{x^{2+2n}\}'\{x^{2r+4n-2}\}'}{\{x^{2r+2n}\}'\{x^{4n}\}'}$$

$$\rho(\xi_j) = C_2 = (x^{2n}; x^{2n})\frac{\{x^{2+2n}\}}{\{x^{2r-2+2n}\}}$$

$$G_1(v) = \frac{\{x^{2+2v}\}'\{x^{2r+2n-2+2v}\}'\{x^{2+2n-2v}\}'\{x^{2r+4n-2-2v}\}'}{\{x^{2r+2v}\}'\{x^{2n+2v}\}'\{x^{2r+2n-2v}\}'\{x^{4n-2v}\}'}$$

$$G_{n-1}(v) = \frac{\{x^{n+2v}\}'\{x^{2r+n+2v}\}'\{x^{3n-2v}\}'\{x^{2r+3n-2v}\}'}{\{x^{2r+n-2+2v}\}'\{x^{2+n+2v}\}'\{x^{2r+3n-2-2v}\}'\{x^{3n+2-2v}\}'}$$

$$S(v) = \frac{\{x^{2r-1+2v}\}\{x^{2r-1+2n-2v}\}}{\{x^{1+2v}\}\{x^{1+2n-2v}\}}$$

$$T(v) = (x^{2v}; x^{2n})(x^{2n-2v}; x^{2n})\frac{\{x^{2+2v}\}\{x^{2+2n-2v}\}}{\{x^{2r-2+2v}\}\{x^{2r-2+2n-2v}\}}.$$

For all other combinations of $\eta_1(v), \eta_{n-1}(v), \xi_j(v)$, we have $\langle\langle XY\rangle\rangle = \rho(X)\rho(Y)$ and $\rho(\eta_j) = C_1\rho(\xi_j) = C_2$.

We only consider the type I vertex operators [3]. We can further construct the bosonization for type II vertex operators of the Z_n-symmetric Belavin model.

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